A Central Difference Numerical Scheme for Fractional Optimal Control Problems

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Abstract

This paper presents a modified numerical scheme for a class of Fractional Optimal Control Problems (FOCPs) formulated in Agrawal (2004) where a Fractional Derivative (FD) is defined in the Riemann-Liouville sense. In this scheme, the entire time domain is divided into several subdomains, and a fractional derivative (FDs) at a time node point is approximated using a modified Grünwald-Letnikov approach. For the first order derivative, the proposed modified Grünwald-Letnikov definition leads to a central difference scheme. When the approximations are substituted into the Fractional Optimal Control (FCO) equations, it leads to a set of algebraic equations which are solved using a direct numerical technique. Two examples, one time-invariant and the other time-variant, are considered to study the performance of the numerical scheme. Results show that 1) as the order of the derivative approaches an integer value, these formulations lead to solutions for integer order system, and 2) as the sizes of the subdomains are reduced, the solutions converge. It is hoped that the present scheme would lead to stable numerical methods for fractional differential equations and optimal control problems.

Keywords: Fractional calculus, Riemann-Liouville fractional derivatives, modified Grünwald-Letnikov approximation, fractional optimal control

1 Introduction

Optimal Control Problems (OCPs) appear in engineering, science, economics, and many other fields. An extensive body of work exists in the area of optimal control of integer order dynamic systems (Hestenes (1966), Bryson and Ho(1975),

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Gregory and Lin(1992)). It was shown recently that fractional derivatives provide more accurate behavior of a dynamic system (see Podlubny (1999) and the references there in). Therefore, formulations and numerical schemes for optimal control problems which account for fractional dynamics of these systems would be necessary. In this work, we develop a modified numerical scheme for a class of Fractional Optimal Control Problems whose dynamics is described by Fractional Differential Equations.

Agrawal (Agrawal (2004)) defines a *Fractional Dynamic System* (FDS) as a system whose dynamics is described by *Fractional Differential Equations* (FDEs), and a *Fractional Optimal Control Problem* (FOCP) as an optimal control problem for an FDS. A general formulation for FOCPs was proposed in Agrawal (2004). As it can be seen from literature, there is no much work in the field of optimal control of FDSs. The formulations of FOCPs comes from Fractional Variational Calculus (FVC) which is an emerging branch of fractional calculus.

Riewe (Riewe (1996), Riewe (1997)) was first to formulate a fractional variational mechanics problem. Riewe's major focus was to develop Lagrangian and Hamiltonian mechanics for dissipative systems. Agrawal (Agrawal (2001)) presented an ad hoc approach to obtain the differential equations of fractionally damped systems. In Agrawal (2002), Agrawal presented fractional Euler-Lagrange equations for Fractional Variational Problems (FVPs). Klimek (Klimek (2001)) presented a fractional sequential mechanics model with symmetric fractional derivatives. In Klimek (2002), Klimek presented stationary conservation laws for fractional differential equations with variable coefficients. Dreisigmeyer and Young (2003) presented nonconservative Lagrangian mechanics using a generalized function approach. In Dreisigmeyer and Young(2004) the authors show that obtaining differential equations for a nonconservative system using FDs may not be possible.

The fractional Euler-Lagrange equation has recently been used by Baleanu and coworker to model fractional Lagrangian with linear velocities (Baleanu and Avkar(2004)), fractional metafluid dynamics (Baleanu (2004)), fractional Lagrangian and Hamiltonian formulations of discrete and continuous systems (Muslih and Baleanu (2005a), Muslih and Baleanu (2005b), Muslih et al.(2006), Baleanu and Muslih(2005)) and Hamiltonian analysis of irregular systems (Baleanu (2006)). Tarasov and Zaslavsky have used variational Euler-Lagrange equation to derive fractional generalization of the Ginzburg-Landau equation for fractal media (Tarasov and Zaslavsky(2005)) and dynamic systems subjected to nonholonomic constraints (Tarasov and Zaslavsky(2006)). In (Agrawal (2004), Agrawal (2005)), the fractional variational calculus is applied to deterministic and stochastic analysis of fractional optimal control problems. Rabei, Ajlouni and Ghassib (2006) develop suitable Lagrangian and Hamiltonian for a fractional dynamic system, which they transform to fractional Schrodinger's equation and solve it. Stanislavsky (2006) presents a Hamiltonian formulation of a dynamic system. Atanackovic and Stankovic (2007) present existence and uniqueness criteria for

problems resulting from fractional variational calculus.

In this paper, we present a direct numerical scheme for a class of Fractional Optimal Control Problems (FOCPs) formulated in Agrawal (2004). The scheme uses a modified Grünwald-Letnikov definition to approximate a fractional derivative. For a first order derivative, this approximation leads to a central difference formula. For simplicity in the discussion to follow, this formulation is briefly presented here. Two examples are solved to demonstrate the performance of the algorithm.

2 Fractional Optimal Control Formulation

In this section, we briefly present a Hamiltonian formulation for an FOCP. Consider the following FOCP: Find the optimal control $u(t)$ for a FDS that minimizes the performance index

$$J(u) = \int_0^1 f(x,u,t)dt \tag{1}$$

and satisfies the system dynamic constraints

$$_0D_t^\alpha x = g(x,u,t), \tag{2}$$

and the initial condition

$$x(0) = x_0, \tag{3}$$

where $x(t)$ is the state variable, t represents the time, f and g are two arbitrary functions, and $_0D_t^\alpha x$ represents the left Riemann-Liouville derivative of order α of x with respect to t. For the definitions of fractional derivatives and some of their applications, see (Podlubny (1999), Magin (2006), and Kilbas, Srivastava and Trujillo (2006)). Note that the upper limit of the integration is taken as 1. We consider $0 < \alpha < 1$. Further, we consider that $x(t)$, $u(t)$, $f(x,u,t)$ and $g(x,u,t)$ are all scalar functions. These conditions are made for simplicity. The same procedure could be followed if the upper limit of integration and α are greater than 1, and $x(t)$, $u(t)$, $f(x,u,t)$ and $g(x,u,t)$ are vector functions.

It should be pointed out that in traditional integer-order optimal Control, Eq. (1) may also include terminal terms. Such terms lead to nonzero terminal condition at $t = 1$. For the FOCP considered here, our formulation would require fractional terminal terms, the meaning of which may not be clear. For this reason, the terminal terms are not included in Eq. (1).

To find the optimal control we define a modified performance index as

$$\bar{J}(u) = \int_0^1 [H(x,u,t) - \lambda\, {_0D_t^\alpha} x]dt, \tag{4}$$

where $H(x,u,\lambda,t)$ is the Hamiltonian of the system defined as

$$H(x,u,\lambda,t) = f(x,u,t) + \lambda g(x,u,t), \tag{5}$$

and λ is the Lagrange multiplier. Taking variations of Eq. (4) and using (5), the necessary equations for the optimal control are given as

$$ {}_tD_1^\alpha \lambda = \frac{\partial H}{\partial x}, \tag{6} $$

$$ \frac{\partial H}{\partial u} = 0, \tag{7} $$

and

$$ {}_0D_t^\alpha x = \frac{\partial H}{\partial \lambda} \tag{8} $$

Following the approach presented in Agrawal (2004), we also require that

$$ \lambda(1) = 0. \tag{9} $$

Equations (6)-(9) represent the necessary conditions in terms of a Hamiltonian for the optimal control of the FOCP defined above. It could be verified that the total time derivative of the Hamiltonian as defined above is not zero along the optimum trajectory even when f and g do not explicitly depend on t. This is a departure from the integer order optimal control theory.

In the discussion to follow, we shall strictly focus on the following quadratic performance index

$$ J(u) = \frac{1}{2}\int_0^1 [q(t)x^2(t) + r(t)u^2]dt, \tag{10} $$

where $q(t) \geq 0$ and $r(t) > 0$, and the system whose dynamics is described by the following linear FDE,

$$ {}_0D_t^\alpha x = a(t)x + b(t)u. \tag{11} $$

Using Eqs. (6) to (8), it can be demonstrated that the necessary Euler-Lagrange equations for this system are (see also Agrawal (2004)),

$$ {}_0D_t^\alpha x = a(t)x - r^{-1}(t)b^2(t)\lambda, \tag{12} $$

$$ {}_tD_1^\alpha \lambda = q(t)x + a(t)\lambda, \tag{13} $$

and

$$ u = -r^{-1}(t)b(t)\lambda. \tag{14} $$

Equations (12) to (14) will be used to develop a direct numerical scheme for a FOCP.

3 A Modified Numerical Scheme for FOCPs

In this section, we define modified Grünwald-Letnikov approximations of fractional derivatives as

$$ {}_0\mathbf{D}_t^\alpha x(t_{i-1/2}) \cong \frac{1}{h^\alpha}\sum_{j=0}^{i} \omega_j^{(\alpha)} x_{i-j}, \qquad i = 1, \cdots, n. \tag{15} $$

$$
{}_t\mathbf{D}_1^\alpha u(t_{i+1/2}) \cong \frac{1}{h^\alpha}\sum_{j=0}^{n-i}\omega_j^{(\alpha)} u_{i+j}, \quad i = n-1, n-2, \cdots, 0, \tag{16}
$$

where $\omega_j^{(\alpha)}$, $j = 0, 1, \cdots, n$ are the coefficients. A recursive approach of computing $\omega_j^{(\alpha)}$ is given as (Podlubny (1999))

$$
\omega_0^{(\alpha)} = 1, \quad \omega_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right)\omega_{j-1}^{(\alpha)}, \quad j = 1, \cdots, n.
$$

It can be shown that for $\alpha = 1$, Eqs. (15) and (16) lead to

$$
\frac{dx(t_{i-1/2})}{dt} = \frac{x_i - x_{i-1}}{h}
$$

and

$$
-\frac{dx(t_{i+1/2})}{dt} = \frac{x_i - x_{i+1}}{h}
$$

which are essentially the central difference equations for the left and the right derivatives.

To develop a numerical scheme, we divide the time domain [0, 1] into n equal parts, and approximate the fractional derivatives ${}_0D_t^\alpha x$ and ${}_tD_1^\alpha \lambda$ at the center of each part using Eqs. (15) and (16). We further take $x(t_{i-1/2})$ as an average of the two end values of the segment. Thus, $x(t_{i-1/2}) = (x_{i-1} + x_i)/2$. We make similar approximations for $\lambda(t_{i-1/2})$, $x(t_{i+1/2})$, and $\lambda(t_{i+1/2})$. Substituting these approximations into (12) and (13), we obtain

$$
\frac{1}{h^\alpha}\sum_{j=0}^{i}\omega_j^{(\alpha)} x_{i-j} = \frac{1}{2}a(i_1h)(x_{i-1} + x_i) - \frac{1}{2}r^{-1}(i_1h)b^2(i_1h)(\lambda_{i-1} + \lambda_i)
$$

$$
i = 1, \cdots, n \tag{17}
$$

$$
\frac{1}{h^\alpha}\sum_{j=0}^{n-i}\omega_j^{(\alpha)} \lambda_{i+j} = \frac{1}{2}q(i_2h)(x_{i+1} + x_i) + \frac{1}{2}a(i_2h)(\lambda_{i-1} + \lambda_i)
$$

$$
i = n-1, \cdots, 0 \tag{18}
$$

where $i_1 = i - \frac{1}{2}$ and $i_2 = i + \frac{1}{2}$. Equations (17) and (18) represent a set of $2n$ linear equations in terms of $2n$ unknowns, which can be solved using a standard linear solver. One can also develop an iterative scheme in which one can march forward to compute x_i's and backward to compute λ_i's to save storage space and perhaps computational time.

4 Numerical Examples

To demonstrate the applicability of the formulation and to validate the numerical scheme, in this section we present numerical results for two problems, time invariant and time varying. For both problems, two types of studies were conducted. The first study involved examination of the response as the number of divisions was increased. For this purpose, N was taken as 8, 16, 32, 64, 128 and 256. The second study involved examination of the response as the order of derivatives approach 1. Results of these studies are given below.

4.1 Time Invariant FOCP

As a first example, we consider the following Time Invariant Problem (TIP): Find the control $u(t)$ which minimizes the quadratic performance index

$$J(u) = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]dt \tag{19}$$

subjected to the system dynamics

$${}_0D_t^\alpha x = -x + u, \tag{20}$$

and the initial condition

$$x(0) = 1. \tag{21}$$

For this example, we have

$$q(t) = r(t) = -a(t) = b(t) = x_0 = 1. \tag{22}$$

This example is considered here because, for $\alpha = 1$, it is one of the most common examples of time invariant systems considered by many. The closed form solution for this system for $\alpha = 1$ is given as (see, Agrawal (1989))

$$x(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t) \tag{23}$$

and

$$u(t) = (1 + \sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2} + \beta)\sinh(\sqrt{2}t) \tag{24}$$

where

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})} \approx -0.9799. \tag{25}$$

From Eqs. (24) and (25), we get $u(0) = -0.3858$.

Figures 1 and 2 show the state $x(t)$ and the control $u(t)$ as functions of t for $\alpha = 0.75$ and different values of N.

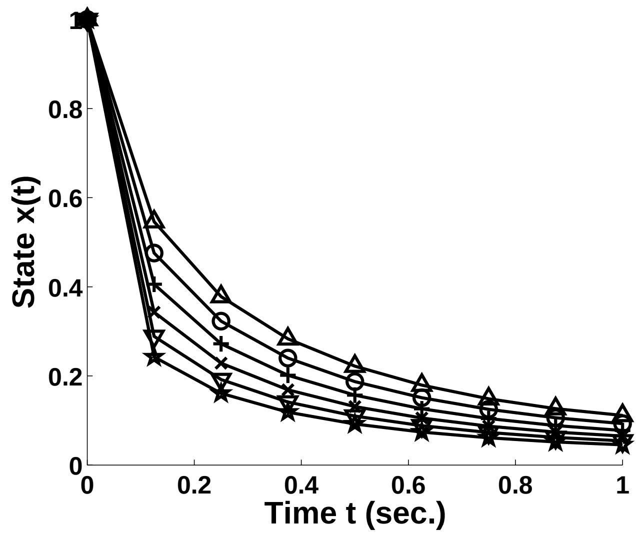

Figure 1: Convergence of $x(t)$ for the TIP for $\alpha = 0.75$ ($\Delta : N = 8$, $O : N = 16$, $+ : N = 32$, $X : N = 64$, $\nabla : N = 128$, $\star : N = 256$)

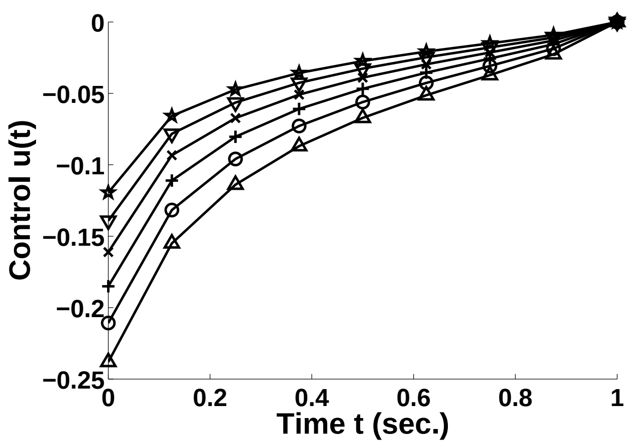

Figure 2: Convergence of $u(t)$ for the TIP for $\alpha = 0.75$ ($\Delta : N = 8$, $O : N = 16$, $+ : N = 32$, $X : N = 64$, $\nabla : N = 128$, $\star : N = 256$)

Figures 3 and 4 show the state $x(1)$ and the control $u(0)$ as a function of N for different α. From these figures, it can be seen that the solutions converge as N is increased, however, the convergence is slow. Further, the convergence becomes poor as α is decreased. Further error analysis may be necessary to identify the reasons for this behavior.

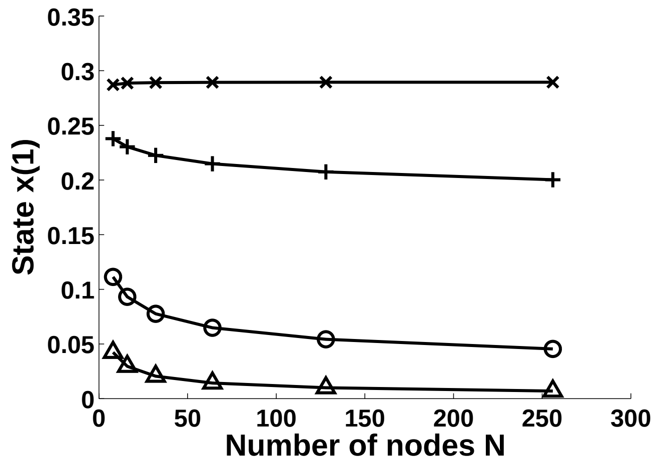

Figure 3: Convergence of $x(1)$ for the TIP for different α ($\Delta : \alpha = 0.5$, $O : \alpha = 0.75$, $+ : \alpha = 0.95$, $X : \alpha = 1.0$)

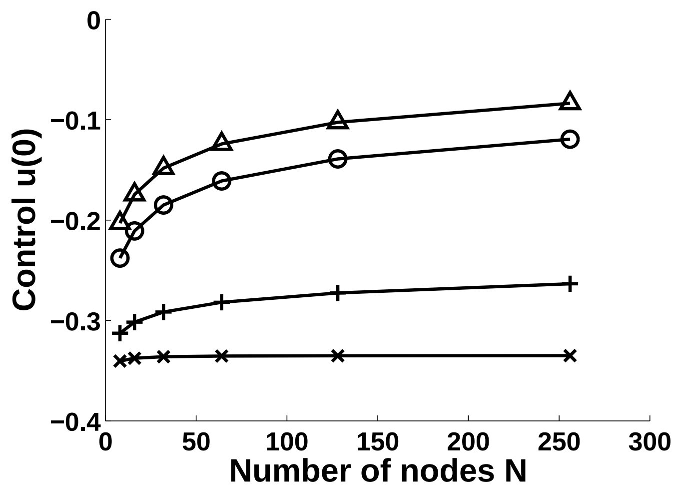

Figure 4: Convergence of $u(0)$ for the TIP for different α ($\Delta : \alpha = 0.5$, $O : \alpha = 0.75$, $+ : \alpha = 0.95$, $X : \alpha = 1.0$)

Figures 5 and 6 show the state $x(t)$ and the control $u(t)$ as functions of t for different values of α. These figures also show analytical results for the state $x(t)$ and the control $u(t)$ for $\alpha = 1$. It can be observed that for $\alpha = 1$ the numerical solution agrees with the analytical solution. Thus, as α approaches to 1, the solution for the integer order system is recovered.

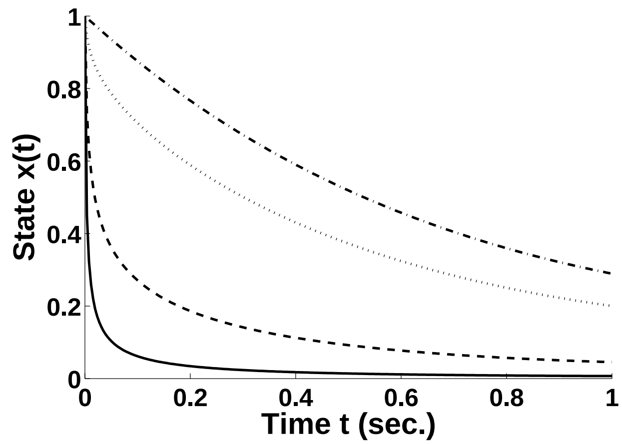

Figure 5: State $x(t)$ as a function of t for the TIP for different α
($-$:$\alpha = 0.5$, --- :$\alpha = 0.75$, $\cdots$:$\alpha = 0.95$, -·-·- :$\alpha = 1$)

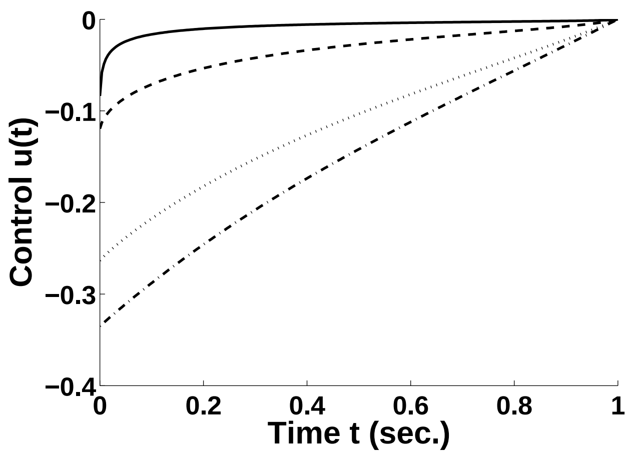

Figure 6: Control $u(t)$ as a function of t for the TIP for different α
($-$:$\alpha = 0.5$, --- :$\alpha = 0.75$, $\cdots$:$\alpha = 0.95$, -·-·- :$\alpha = 1$)

4.2 Time Varying FOCP

As a second example, we consider the following Time Varying Problem (TVP): Find the control $u(t)$ which minimizes the quadratic performance index given in Eq. (19), and which satisfies the system dynamics

$$ {}_0D_t^\alpha x = tx + u. \tag{26} $$

The initial condition is $x(0) = 1$. For this example, we have

$$ q(t) = r(t) = b(t) = x_0 = 1, \quad a(t) = t. \tag{27} $$

It is one of the simplest examples of time varying systems, and for $\alpha = 1$, it has been considered at several other places (see, Agrawal (1989), and the references there in).

Figures 7 and 8 show the state $x(t)$ and the control $u(t)$ as functions of t for different values of N. Figures 9 and 10 show the state $x(1)$ and the control $u(0)$ as a function of N for different α. As for the TIP, the solutions for the TVP also converge as N is increased, however, as before, the convergence is slow. This slow convergence for both examples clearly suggests that further improvement of the scheme is necessary. Figures 11 and 12 show the state $x(t)$ and the control $u(t)$ as functions of t for different values of α.

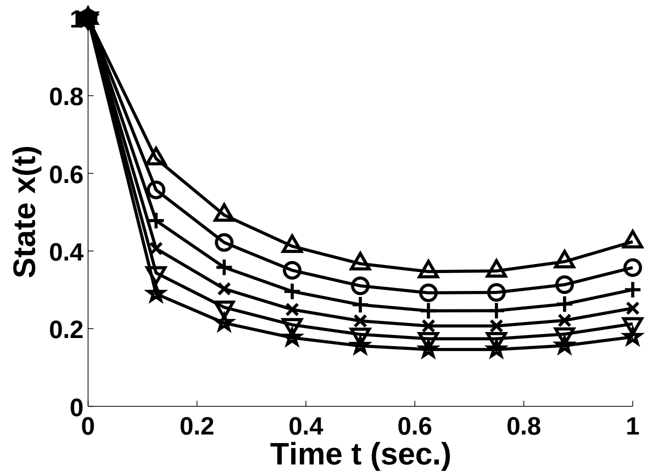

Figure 7: Convergence of $x(t)$ for the TVP for $\alpha = 0.75$ ($\Delta : N = 8$, $O : N = 16$, $+ : N = 32$, $X : N = 64$, $\nabla : N = 128$, $\star : N = 256$)

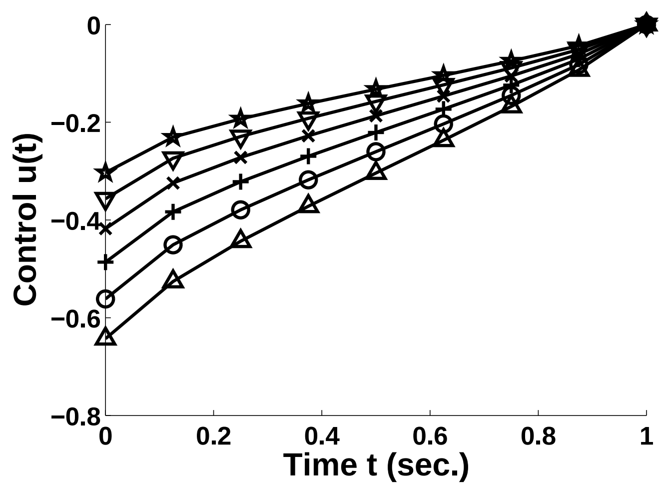

Figure 8: Convergence of $u(t)$ for the TVP for $\alpha = 0.75$ ($\Delta : N = 8$, $O : N = 16$, $+ : N = 32$, $X : N = 64$, $\nabla : N = 128$, $\star : N = 256$)

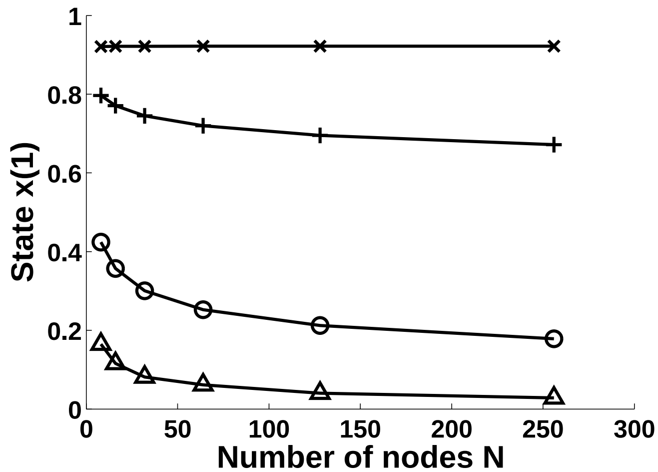

Figure 9: Convergence of $x(1)$ for the TVP for different α ($\Delta : \alpha = 0.5$, $O : \alpha = 0.75$, $+ : \alpha = 0.95$, $X : \alpha = 1.0$)

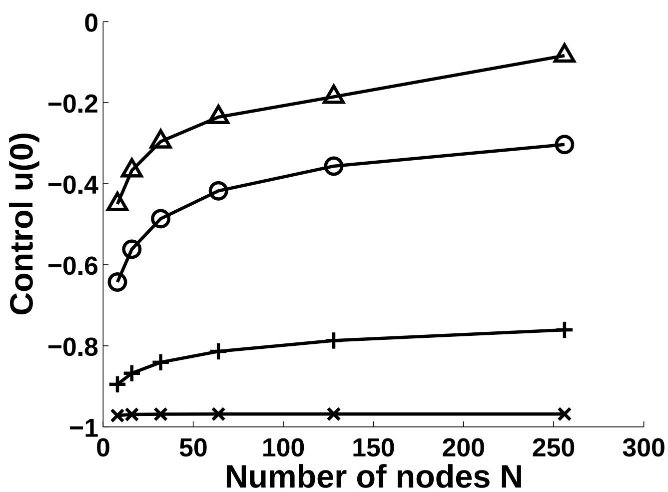

Figure 10: Convergence of $u(0)$ for the TVP for different α ($\Delta : \alpha = 0.5$, $O : \alpha = 0.75$, $+ : \alpha = 0.95$, $X : \alpha = 1.0$)

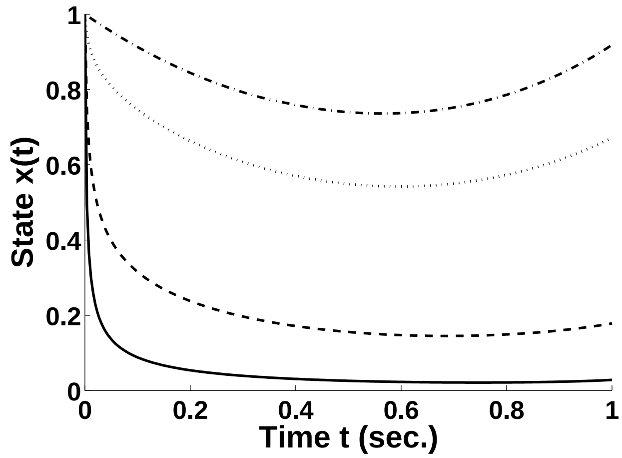

Figure 11: State $x(t)$ as a function of t for the TVP for different α
($-$:$\alpha = 0.5$, --- :$\alpha = 0.75$, ··· :$\alpha = 0.95$,
-·-·- :$\alpha = 1$)

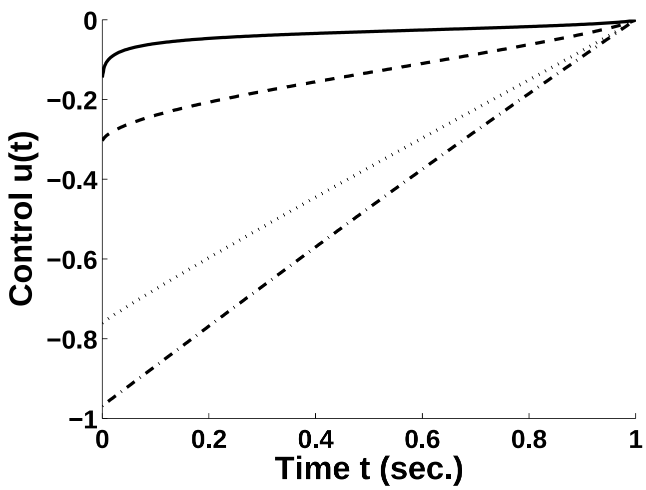

Figure 12: Control $u(t)$ as a function of t for the TVP for different α
($-$:$\alpha = 0.5$, --- :$\alpha = 0.75$, ··· :$\alpha = 0.95$, -·-·- :$\alpha = 1$)

This problem for $\alpha = 1$ has been solved in Agrawal (1989) using a different scheme.The scheme is based on the approximation with weighing coefficients and the lagrange multiplier technique for a class of optimal control problems (Agrawal (1989)). Results show that for $\alpha = 1$ the numerical solutions obtained using the scheme developed here and in Agrawal (1989) agree well. Thus, as before, as α approaches to 1, the solution for the integer order system is recovered.

It should be point out here that in integer order calculus central difference schemes have been used in many cases to develop numerically stable and efficient schemes. It is hoped that this research will initiate a similar effort in fractional calculus.

5 Conclusions

For a general class of fractional optimal control problems a Hamiltonian was defined and a set of necessary conditions were derived. A direct numerical scheme was presented for solution of the problems. The scheme was used to solve two problems, time invariant and time varying. Results showed that as the number of divisions of the time domain was increased, the solutions converged. However, the convergence appears to be slow. As the value of α approaches 1, the solution for the integer order system is recovered. It is hoped that this research would initiate further research in the field, and more efficient and stable schemes would be found.

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